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Steady adiabatic state: Its thermodynamics, entropy production, energy dissipation, and violation of Onsager relations

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A class of statistical systems is considered where different degrees of freedom have well-separated characteristic times, and are described by different temperatures. The stationary state is a nonequilibrium state with a heat flow. A generalized statistical thermodynamics is constructed and a universal variational principle is proposed. Entropy production and energy dissipation occur at a constant rate. To leading order in the small ratio of the characteristic times, there exists a universal relation between them. Onsager relations in the context of heat transfer are also considered. They are always broken, except close to equilibrium.

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I. INTRODUCTION

Statistical thermodynamics is a powerful theory for describing equilibrium states [1,2]. Due to the general character of the Gibbs distribution, thermodynamics produces universal relations, which apply independently of the detailed behavior. It has been generalized to weakly nonequilibrium states in an approach first started by Onsager, and further developed extensively (see, e.g., [1–5]). In this paper we propose a thermodynamical description for a class of situations far from equilibrium: the steady adiabatic state. Having a stationary distribution far from Gibbsian, these systems possess widely separated characteristic times (adiabaticity) that appear enough to deduce a thermodynamical description.

A frequently encountered generalization of the usual equilibrium is a system interacting with different thermal baths at constant temperatures [6]. In the infinite-time limit the system goes to a nonequilibrium steady state with heat currents between the baths. In contrast to the Gibbs distribution, the steady state depends on the underlying time scales, and still contains information about the dynamics. Therefore, we call it *the steady adiabatic state*. The system has been studied as one of the minimal models having easily controlled nonequilibrium properties [6]. The conceptual and technical advantages of this approach were stressed, in particular, by Meixner [5].

Our investigation focuses on the large separation between the characteristic times. On the other hand, we shall specify neither the Hamiltonian of the system, nor the difference between temperatures. In that sense the results will be universal.

Our purposes are the following: (i) To derive the steady statistical distribution and the corresponding thermodynamics; (ii) to propose a general variational principle for this thermodynamics; (iii) to investigate the important dissipative characteristics, such as entropy production and energy dissipation, in the steady state; and (iv) to show the breakdown of the *nonequilibrium* Onsager relations for heat transfer.

II. THE MODEL AND ITS THERMODYNAMICS

To underline our conclusions, let us introduce a pair of coupled stochastic variables x_1,x_2 having Hamiltonian $H(x_1,x_2)$, which interact with different thermal baths at different characteristic time scales. The Langevin equations read

$$\Gamma_i\dot{x}_i=-\partial_i H(x_1,x_2)+\eta_i(t)\quad (i,j=1,2),$$
$$\langle\eta_i(t)\eta_j(t')\rangle=2\Gamma_i T_i\delta_{ij}\delta(t-t'),\tag{1}$$

where Γ_1,Γ_2 are the damping constants, and $\partial_i=\partial/\partial x_i$. The Einstein relation between the strength of noise and the damping constant holds in Eq. (1), because the thermal baths themselves are assumed to be in equilibrium [1–4,6]. The Fokker-Planck equation that corresponds to Eqs. (1) reads

$$\partial_t P(x_1,x_2;t)+\sum_{i=1}^{2}\partial_i J_i(x_1,x_2;t)=0,\tag{2}$$

$$J_i=-\frac{1}{\Gamma_i}P(x_1,x_2;t)\partial_i H(x_1,x_2)-\frac{T_i}{\Gamma_i}\partial_i P(x_1,x_2;t),\tag{3}$$

where $P(x_1,x_2;t)$ is the probability distribution, and J_1,J_2 are the currents of probability.

We shall assume that x_2 is changing much more slowly than x_1; this is ensured by the condition $\gamma=\Gamma_1/\Gamma_2\ll 1$. Let us first indicate how the stationary distribution can be obtained to order γ^0; this will give us the basic formulation of the generalized thermodynamics. Equations (1) can be investigated by the method of adiabatic elimination [7] (Born-Oppenheimer method). First Eq. (1) for x_1 is solved keeping the x_2 fixed, which is valid on the relatively short time scales where only Eq. (1) for x_1 is relevant. In this limit the Langevin equation for $i=1$ has the obvious equilibrium distribution

$$P_0(x_1|x_2)=\frac{1}{Z(x_2)}\exp[-\beta_1 H(x_1,x_2)],\tag{4}$$

where $Z(x_2)$ is the partition sum for a fixed value of x_2. At quasiequilibrium of the x_2 subsystem this average should be performed using the distribution (4). In this way we get from Eqs. (1) a related dynamics for the slow variable, in which the two particle Hamiltonian $H(x_1,x_2)$ is replaced by the effective one-particle Hamiltonian $H_e(x_2)$,

$$H_e(x_2)=-T_1\ln Z(x_2). \tag{5}$$

We thus have the effective equation of motion

$$\Gamma_2\dot{x}_2=\frac{\partial}{\partial x_2}T_1\ln Z(x_2)+\eta_2(t), \tag{6}$$

and the corresponding Fokker-Planck equation

$$\partial_t P(x_2,t)=\frac{1}{\Gamma_2}\partial_2[P(x_2,t)\partial_2 H_e(x_2)+T_2\partial_2 P(x_2,t)]. \tag{7}$$

As the noise is due to a bath at temperature T_2 [see Eq. (1)], the equilibrium distribution of this process reads

$$P_0(x_2)=\frac{Z^{T_1/T_2}(x_2)}{\mathcal{Z}},\quad \mathcal{Z}=\int dx_2 Z^{T_1/T_2}(x_2). \tag{8}$$

The joint distribution of x_1 and x_2 can now be written as

$$P_0(x_1,x_2)=P_0(x_1|x_2)P_0(x_2). \tag{9}$$

Let us stress that this steady state is not Gibbsian. It will be so if the coupling between x_1 and x_2 is weak; then the state (9) will be the product of two separate Gibbsian distributions at the temperatures T_1 and T_2. On the other hand, if $T_1=T_2$ one obtains from Eq. (9) the usual Gibbs distribution. A similar approach is applied in spin glasses and other disordered systems where $n=T_1/T_2$ is considered as a ''dynamically generated'' replica number [8].

If the state of a system is described by a distribution $P(x_1,x_2,t)=P(x_1|x_2,t)P(x_2,t)$, then there are general definitions for the mean energy and entropy [4]:

$$U=\int dx_1dx_2P(x_1,x_2,t)H(x_1,x_2), \tag{10}$$

$$S=-\int dx_1dx_2P(x_1,x_2,t)\ln P(x_1,x_2,t). \tag{11}$$

This latter Boltzmann-Gibbs-Shannon formula corresponds to the general statistical definition of entropy, relevant also outside of equilibrium [4]. The total entropy can be decomposed as

$$S=S_1+S_2,$$

$$S_1=\int dx_2\,P(x_2,t)\left(-\int dx_1P(x_1|x_2,t)\ln P(x_1|x_2,t)\right),$$

$$S_2=-\int dx_2P(x_2,t)\ln P(x_2,t), \tag{12}$$

where S_1 is the *relative* entropy [4], namely, the entropy of the fast variable x_1, averaged over the quenched slow variable x_2, and S_2 is the entropy of the slow variable itself. This general result of the statistical thermodynamics can again be applied in previous case where $P(x_1,x_2,t)=P_0(x_1,x_2)$. Now we define the free energy of the two-temperature situation, $F=-T_2\ln\mathcal{Z}$,

$$F=U-T_1S_1-T_2S_2. \tag{13}$$

The entropies S_1,S_2 can be obtained as partial derivatives of F,

$$\left.\frac{\partial F}{\partial T_1}\right|_{T_2}=-S_1,\quad \left.\frac{\partial F}{\partial T_2}\right|_{T_1}=-S_2. \tag{14}$$

Equation (13) agrees with the expression of the free energy for a glassy system put forward previously by one of us, analyzing solvable models [9,10]. In that approach the equivalent of T_2 is the dynamically generated effective temperature, while here it is the temperature of a bath. A similar two-temperature steady state approach occurred recently when we analyzed the so-called p-spin model [11].

The free energy (13) has several essential properties of its equilibrium companion. Let α be some parameter of the Hamiltonian, which is varying externally according to a given trajectory $\alpha(t)$. The change of the mean energy (10) will consist of two terms,

$$dU=\int dx_1dx_2[P(x_1,x_2,t)dH(x_1,x_2,\alpha(t))+H(x_1,x_2,\alpha(t))dP(x_1,x_2,t)], \tag{15}$$

where $P(x_1,x_2,t)$ is the corresponding solution of the Fokker-Planck equation (2). The first term in Eq. (15) is due to the Hamiltonian, which is essentially a mechanistic, nonstatistical object; thus it is associated with the work $đW$ produced by external sources [1,2]. The second term represents the variation due to the statistical redistribution of the configurational space, and is identified with a heat $đQ$. The first law can be written as usual:

$$dU=đW+đQ. \tag{16}$$

Now the characteristic time $\mathcal{T}$ of this variation is assumed to be much larger than the maximal relaxation time Γ_2. As usual, this means that the variation is done by an external macroscopic source [1,2]. Further, we shall consider the isothermal variation, namely, both temperatures T_1,T_2 are fixed. The variation starts at the initial time t_i, and is finished at $t=t_f$. Both these times are also assumed larger than Γ_2, such that the variation starts when the system is already in the steady state. In the zero order of $\Gamma_2/\mathcal{T}$ the time-dependent distribution will be $P_0(x_1,x_2,\alpha(t))$, namely, the steady-state distribution, where $\alpha(t)$ is substituted. This is clear from Eq. (2), where for the slow variation one can neglect the left-hand system. Now it can be shown directly that

$$dW=dF. \tag{17}$$

Thus, the free energy (13) determines the isothermal work done by external sources, when the parameter changes very slowly.

Using Eq. (16) we shall obtain for the heat

$$dQ=T_1dS_1+T_2dS_2. \tag{18}$$

Results similar to Eq. (18) were obtained in investigating solvable glassy systems [9] and noisy electrical circuits [12].

Dissipative effects due to a nonoptimal variation of the parameter can be naturally incorporated into the presented scheme. As a simple but illustrative example, let us assume that $\mathcal{T}$ is still much larger than Γ_2, but remains finite. The first correction to the distribution function $P(x_1,x_2,t)$ will arise due to the deviation of x_2 from its locally stable distribution $P_0(x_2,\alpha(t))$, while the fast degree will still be described by $P_0(x_1|x_2,\alpha(t))$. For simplicity we shall assume a ''smooth'' protocol of variation, where $\dot{\alpha}(t_i)=0$.

Starting from Eq. (7) one can show by direct substitution that the corrected distribution function will be

$$\begin{aligned}P(x_1,x_2,t)&=P(x_2,t)P_0(x_1|x_2,\alpha(t))\\&=P_0(x_1,x_2,\alpha(t))[1+\Gamma_2\beta_2B(x_2,t)-\langle B\rangle_0],\end{aligned} \tag{19}$$

where $\langle\cdots\rangle_0$ means averaging by $P_0(x_1,x_2,\alpha(t))$, and

$$B(x_2,t)=\int_{-\infty}^{x_2}\frac{dy}{P_0(x_2,\alpha(t))}\int_{-\infty}^{y}dz\,\partial_tP_0(z,\alpha(t)). \tag{20}$$

Since

$$\partial_tP_0(z,\alpha(t))=-\dot{\alpha}(t)P_0(z,\alpha(t))[\partial_\alpha H_e(z,\alpha(t))-\langle\partial_\alpha H_e\rangle_0], \tag{21}$$

the correction in Eq. (19) contains the small prefactor $\dot{\alpha}(t)\Gamma_2$, which controls the slow, but not too slow, variation of α. Distribution (19) satisfies the initial condition $P(x_1,x_2,t_i)=P_0(x_1,x_2,\alpha(t_i))$, since $\dot{\alpha}(t_i)=0$.

We shall collect Eqs. (19)–(21) and the definition of work, use the fact that

$$\partial_\alpha H_e(x_2,\alpha)=\int dx_1P_0(x_1|x_2)\partial_\alpha H(x_1,x_2,\alpha), \tag{22}$$

which follows directly from Eq. (5), and get after integration by parts the following expression for the work ΔW done, when varying the parameter α from the initial time t_i to the final time t_f:

$$\Delta W=\Delta F+\Delta\Phi, \tag{23}$$

$$\begin{aligned}\Delta\Phi=\frac{\Gamma_2}{T_2^2}\int_{t_i}^{t_f}dt\,\dot{\alpha}^2(t)\int\frac{dx}{P_0(x,\alpha(t))}\bigg(&\int_x^\infty dy\,P_0(y,\alpha(t))\\&\times[\partial_\alpha H_e(y,\alpha(t))-\langle\partial_\alpha H_e\rangle_0]\bigg)^2\geqslant 0.\end{aligned} \tag{24}$$

$\Delta F=F(\alpha(t_f))-F(\alpha(t_i))$ is the difference between free energies, which determines the purely adiabatic contribution to the work, and $\Delta\Phi$ is the corresponding nonadiabatic part, which is positive, and should be identified with the energy dissipated due to nonoptimal variation of the parameter. Notice in this context that $\Delta\Phi$ depends on the whole trajectory $\alpha(t)$, in contrast to ΔF.

Now Eq. (18) can be written in the form of the generalized Clausius inequality

$$T_1dS_1+T_2dS_2-\text{đ}Q=\text{đ}\Phi\geqslant 0. \tag{25}$$

Equations (13), (14), (17), (18), (23), and (25) are manifestations of the generalized thermodynamics of the steady adiabatic state.

Variational principle

As is well known, variational principles play an important role in thermodynamics, since they provide general methods of derivation of the corresponding stationary probability distributions. Indeed, they can be used *independently* on the underlying dynamics, when no detailed information concerning the dynamics is available [1–5].

Let us indicate how distribution (9) can be obtained from a more general consideration. The usual Gibbs distribution for homogeneous equilibrium states can be obtained either from maximizing the entropy, keeping the energy fixed, or from minimizing the energy, keeping the entropy fixed. For the steady adiabatic state, which is nonhomogeneous and out of equilibrium, one can minimize the mean energy, keeping both entropies S_1 and S_2 fixed. Following the standard method we should minimize the Lagrange function

$$\begin{aligned}\mathcal{L}=\int dx_1dx_2P(x_1,x_2)H+T_2\int dx_2P(x_2)\ln P(x_2)\\+T_1\int dx_2P(x_2)\int dx_1P(x_1|x_2)\ln P(x_1|x_2),\end{aligned} \tag{26}$$

where T_1 and T_2 are Lagrange multipliers and normalize the solutions. In doing the minimization with respect to $P(x_2)$ and $P(x_1|x_2)$, we now recover Eqs. (4) and (8), but this time on the basis of a more general variational principle. In contrast to the usual equilibrium (or weakly nonequilibrium) situation, it is impossible now to interchange the roles of entropy and energy.

For $T_1\neq T_2$ this principle will be the direct consequence of the equilibrium variational principle *only* in the trivial case when the coupling between x_1 and x_2 is negligible, and the average energy is the sum of two partial energies: $U=U_1+U_2$.

III. ENTROPY PRODUCTION AND ENERGY DISSIPATION IN THE STEADY STATE

In the previous section we studied thermodynamic relations when a parameter is varied by external sources. This section is devoted to dissipative characteristics, which are inherently present in the system due to the difference between temperatures T_1 and T_2.

The different temperatures at the steady state lead to a constant heat current through the system. This implies a constant production of entropy and a dissipation of energy. We investigate these effects taking into account possible γ corrections. The stationary probability distribution can be expressed as

$$P_1(x_1,x_2)=P_0(x_1,x_2)[1-\gamma A(x_1,x_2)]+O(\gamma^2). \quad (27)$$

The boundary conditions are, as usual, that $P(x_1,x_2)$ and its derivatives vanish at infinity. $A(x_1,x_2)$ is obtained from the stationarity condition $\partial_t P(x_1,x_2,t)=0$, taking into account the orthogonality condition $\int dx_1 dx_2 A P_0=0$ and consistency with $O(\gamma^2)$ terms. The general expression for A is rather lengthy, but for a concrete model it is given in Eq. (41) below. The steady currents (3) are given by

$$J_1(x_1,x_2)=\frac{T_1}{\Gamma_2}P_0(x_1,x_2)\partial_1 A(x_1,x_2), \quad (28)$$

$$J_2(x_1,x_2)=\frac{T_1-T_2}{T_1\Gamma_2}P_0(x_1,x_2)\,\delta F_2(x_1,x_2). \quad (29)$$

Notice that for J_2 the object A is not needed, but only

$$\delta F_2(x_1,x_2)=-\partial_2 H(x_1,x_2)+\int dy\, P_0(y|x_2)\partial_2 H(y,x_2), \quad (30)$$

which is the difference between the force acting on the second subsystem and its conventional mean value obtained by averaging over the fast degree of freedom. Therefore some of the further results can be derived without knowledge of A, although it is needed for consistency checks and γ corrections.

The change of the total entropy reads

$$\dj S_{tot}=dS+\dj S_{b,1}+\dj S_{b,2}=dS-\beta_1\dj_1 Q-\beta_2\dj_2 Q, \quad (31)$$

where $S=S_1+S_2$ is the entropy of the system defined by Eq. (12), $S_{b,1},S_{b,2}$ are the entropies of the corresponding thermal baths, and $\dj_1 Q,\dj_2 Q$ are the amounts of heat obtained by the system from the thermal baths. Conservation of energy means that

$$\dj_i Q=-\dj Q_{b,i}=-T_i\dj S_{b,i}, \quad (32)$$

where the second equality holds because the baths are in equilibrium. The flow of heat from bath i,

$$\dot{Q}_i\equiv\frac{\dj_i Q}{dt}=-\int dx_1 dx_2 H(x_1,x_2)\partial_i J_i(x_1,x_2,t), \quad (33)$$

can be obtained from Eq. (2). The entropy and the mean energy of the stationary state are constant: $\dot{S}=0,\dot{Q}_1+\dot{Q}_2=0$. Nevertheless, there exists a constant-rate entropy production in the outside world (the thermal baths),

$$\dot{S}_{tot}=(\beta_1-\beta_2)\dot{Q}_2. \quad (34)$$

Energy dissipation in the steady state can be defined as the sum of energies dissipated under the influence of the baths:

$$\dot{\Pi}=\sum_{i=1}^{2}\left(T_i\frac{\dj_i S}{dt}-\frac{\dj_i Q}{dt}\right). \quad (35)$$

Recall that $\dj_i Q$ of Eq. (33) is the heat obtained from the thermal bath i, and $\dj_i S$, defined similarly with $H(x_1,x_2)\to-\ln P(x_1,x_2)$, is the change of entropy induced by this bath. For the adiabatic system considered this definition can be shown to have a more intuitive meaning. Starting from Eqs. (2) and (13), using Eq. (33) and also $\dot{S}_1=-\int\dot{P}(1,2)\ln P(1|2), \dot{S}_2=-\int\dot{P}(1,2)\ln P(2)$, we obtain

$$\dot{F}(t)=(T_1-T_2)\int dx_1 dx_2 J_2(x_1,x_2,t)\partial_2 \ln P(x_1|x_2,t)$$
$$-\int dx_1 dx_2 P(x_1,x_2,t)\sum_{i=1}^{2}\frac{1}{\Gamma_i}[\partial_i H(x_1,x_2)$$
$$+T_i\partial_i \ln P(x_1,x_2,t)]^2. \quad (36)$$

We may write this as $\dot{F}=\dot{W}_{ss}-\dot{\Pi}$, where $\dot{\Pi}\geqslant 0$ is the energy dissipation defined in Eq. (35), and $\dot{W}_{ss}$ can be associated with the work done to keep the system in the nonequilibrium state. In the stationary state one has $\dot{F}=0$, and the work performed must be equal to the dissipated energy.

Using Eqs. (28) and (30) we get after some algebra for the entropy production and energy dissipation at the steady state

$$\dot{S}_{tot}=\gamma\frac{\kappa^2}{T_2\Gamma_1}\langle(\delta F_2)^2\rangle_1+\gamma^2\frac{\kappa}{\Gamma_1}\langle\delta F_2\,\partial_2 A\rangle_0,$$

$$\dot{\Pi}=\gamma\frac{\kappa^2}{\Gamma_1}\langle(\delta F_2)^2\rangle_1+\gamma^2\frac{\kappa(2T_2-T_1)}{\Gamma_1}\langle\delta F_2\partial_2 A\rangle_0, \quad (37)$$

where $\kappa=(T_1-T_2)/T_1$, and $\langle\cdots\rangle_{0(1)}$ means averaging over the distribution $P_{0(1)}$. We observe the relation

$$\dot{\Pi}=T_2\dot{S}_{tot}+O(\gamma^2) \quad (38)$$

valid to leading order in γ. For the usual nonstationary system tending to equilibrium one has the relation $\dot{\Pi}=T\dot{S}_{tot}$ between entropy production and energy dissipation, where T is the temperature of the unique thermal bath. On the other hand, Eq. (38) reflects degradation of energy in the stationary state. This equation also shows that when T_2 is close to zero, the energy dissipation (but not the entropy production) loses its leading term. Apparently this means that energy dissipation, in contrast to entropy production, is closely related to the fluctuations of slow degree that are controlled by T_2.

Let us apply the general results obtained to a simple model. We consider a pair of weakly interacting oscillators with coordinates x_1,x_2 and Hamiltonian $H=\frac{1}{2}ax_1^2+\frac{1}{2}ax_2^2+gx_1^2x_2^2$, where $a>0,g>0$. Very similar models are applied to describe an oscillator with random frequency [7] or some electrical circuits [4]. For simplicity we shall discuss the model keeping only the first nonvanishing order in the small parameter g. The stationary distribution has the form (27), with

$$P_0(x_1|x_2)=\sqrt{\frac{\beta_1 a}{2\pi}}e^{-\beta_1 x_1^2/2}\left(1-g\beta_1 x_1^2 x_2^2+g\frac{x_2^2}{a}\right), \quad (39)$$

$$P_0(x_2)=\sqrt{\frac{\beta_2 a}{2\pi}}\,e^{-\beta_2 x_2^2/2}\left(1-g\,\frac{T_1\beta_2 x_2^2}{a}+g\,\frac{T_1}{a^2}\right) \tag{40}$$

$$A=\frac{g(T_1-T_2)}{a^2}(1-a\beta_2 x_2^2)(1-a\beta_1 x_1^2). \tag{41}$$

After some calculations we get from Eqs. (28) and (37)

$$\dot{S}_{tot}=\frac{8g^2}{a^3\Gamma_1}(T_1-T_2)^2(\gamma-\gamma^2), \tag{42}$$

$$\dot{\Pi}=\frac{8g^2}{a^3\Gamma_1}(T_1-T_2)^2[\gamma T_2+\gamma^2(T_1-2T_2)]. \tag{43}$$

IV. BREAKDOWN OF ONSAGER RELATIONS

The results obtained allow us finally to discuss the Onsager relations concerning heat transfer. These fundamental and experimentally testable relations were proposed by Onsager to describe transport in weakly nonequilibrium systems (the linear case) [4,13]. Later they were generalized to the nonlinear regime. Following standard arguments [4,13] the Onsager relation reads in our case

$$\partial_{\beta_1}\dot{Q}_2=\partial_{\beta_2}\dot{Q}_1, \tag{44}$$

where $\dot{Q}_i$, given by Eqs. (33) and (34) is the heat flux from the thermal bath *i*. In the stationary case one has $\dot{Q}_1+\dot{Q}_2=0$. For our model we use (34) to express $\dot{Q}_2$ through $\dot{S}_{tot}$, and obtain from Eq. (42)

$$\dot{Q}_2=\frac{8\gamma g^2}{a^3\Gamma_1}\,\frac{\beta_1-\beta_2}{\beta_1^2\beta_2^2}+O(\gamma^2). \tag{45}$$

The linear case corresponds to Eq. (44) with $\beta_1\approx\beta_2$. Indeed, then the fluxes can be written in more familiar form as

$$\dot{Q}_i=\sum_j L_{ij}\Delta\beta_j, \tag{46}$$

where $\Delta\beta_i=\beta_i-\beta_0$ is a small deviation of the inverse temperature β_i from its equilibrium value β_0, and the L_{ij} depend only on β_0 but not on $\beta_{1,2}$ separately. In that case the relation (44) takes the form $L_{12}=L_{21}$ $(=8\gamma g^2T_0^4/a^3\Gamma_1$ in our model). This form of the Onsager relations is applicable mainly for the linear case [4,13], in contrast to the more general relation (44).

One of the arguments leading to Eq. (44) is an *assumption* about the existence of a dissipative potential $\mathcal{F}(\beta_1,\beta_2)$, which generates heat currents

$$\dot{Q}_1=\partial_{\beta_1}\mathcal{F},\quad \dot{Q}_2=\partial_{\beta_2}\mathcal{F}. \tag{47}$$

Then Eq. (44) follows as a simple consequence. As is well known, dissipative potential exists in the linear regime, and coincides there with entropy production [1–5]. Since the validity of Eq. (44) in the linear regime is a fundamental theorem connected with the time-inversion invariance of the equilibrium state [4,2,3], any breaking of Eq. (44) can be connected only with $T_1\neq T_2$. The converse is not true: there are physically important cases where the Onsager relations hold out of equilibrium [4,13]. In particular, in Ref. [14] it was shown that for a special class of nonequilibrium systems the time-inversion invariance can be broken, but Onsager relations are still valid due to an additional symmetry. Thus checking these relations for our concrete class of nonequilibrium systems seems important. From Eqs. (34) and (37), and using $\dot{Q}_1=-\dot{Q}_2$, we obtain in the general situation to leading order in γ

$$\partial_{\beta_1}\dot{Q}_2-\partial_{\beta_2}\dot{Q}_1=\gamma\,T_2\,\frac{\beta_1-\beta_2}{\Gamma_1}\{\partial_{\beta_1}+\partial_{\beta_2}\}\langle(\delta F_2)^2\rangle, \tag{48}$$

In the linear regime with $\beta_1\approx\beta_2$ the Onsager relation is thus satisfied. However, for the considered system that is the exceptional case, and (44) cannot be true in general. Indeed, for our model we get

$$\partial_{\beta_1}\dot{Q}_2-\partial_{\beta_2}\dot{Q}_1=\gamma\,\frac{16g^2}{a^3\Gamma_1}T_1T_2(T_1^2-T_2^2). \tag{49}$$

implying a violation of the Onsager relation for any $T_1\neq T_2$, because, due to Eq. (45), the right-hand side of Eq. (49) has the same order of magnitude as the individual terms in the left-hand side. In this sense the violation is strong. In general, if Eq. (44) were to be valid, Eq. (48) says that we should have $\langle(\delta F_2)^2\rangle=\beta_2 f(\beta_1-\beta_2)$ for all β_1,β_2, where f is some positive function. This cannot hold, since on taking the limit $\beta_2\rightarrow 0$ one obtains zero on the right-hand side, while the left-hand side typically diverges, or at least stays finite and nonzero, including nontypical cases. Thus, the Onsager relations are broken for the class of nonequilibrium systems considered.

V. DISCUSSION

We have considered a class of stochastic systems where two subsystems have well-separated time scales, and interact with different thermal baths. In addition to its direct physical interest, our approach can have certain interesting analogies with glasses, which we wish to suggest for further consideration.

Glasses are weakly time-dependent nonequilibrium systems [15], and when cooling at a proper rate (varying from 10^{-2} K/s for window glass to 10^5 K/s for metallic glasses and 10^{11} K/s for computer glasses) relaxation times become very large near the experimentally defined glassy temperature T_g. The metastable state thus reached slowly approaches equilibrium. For a certain class of systems, namely, those having degrees of freedom with two well-separated characteristic times, the nonequilibrium state can be described by assigning different temperatures (so-called effective or fictive temperatures) [15,16,9,10]. In a certain time window these two-temperature models allow a quasistatic description in which the slow modes appear to be coupled to a real heat bath at the present fictive temperature. Indeed, on time scales of the order of the characteristic time t_{obs} the fictive temperature remains constant. A typical decay is of the form $1/\ln(t_{\mathrm{obs}}/t_1)$ [9]; in the long time regime $t_{\mathrm{obs}} \gg t_1$, this hardly changes for times that are large but still of the order of t_{obs}. The generalized thermodynamics presented by us in Sec. II has clear analogies with glassy thermodynamics [10,9]. In spite of the fact that the second temperature is not self-generated by the system, but is introduced by the real thermal bath, one can expect that many *thermodynamic* relations do not depend on the concrete process that generates this temperature. Further steps in this direction were made in Refs. [11,17], where this analogy was investigated in the context of fluctuation-dissipation relations and response functions.

To conclude, we shall briefly reiterate the main results of the present paper. We consider a class of steady systems with fast and slow processes coupled to two heat baths at a different temperature. The situation is far from equilibrium, but it nevertheless allows a thermodynamic description. The generalized thermodynamics can be obtained by minimization of the energy, keeping all entropies fixed. Universal relations, Eqs. (37) and (38), are obtained between entropy production and energy dissipation. Energy dissipation (in contrast to entropy production) loses its leading term when the temperature of the bath with the slow degrees of freedom goes to zero. Finally, we show that the nonlinear Onsager relation for heat transfer in the steady adiabatic state is always broken, reflecting the system's strongly nonequilibrium character. As the effect is of order unity, it can be testable experimentally.

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